§5.4 Work

Key points:

① Work = Force × Distance, $W = F \cdot d$, (work done by constant force)

② $W = \int_a^b f(x) \cdot dx$ (Work done by Force $f(x)$ from a to b)

☆ ③ Water-Pumping formula: $\text{Work} = \int_a^b \sigma \cdot s(y) \cdot A(y) \cdot dy$

(Two extra physical formulas:)

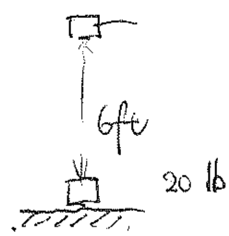
Newton's II Law: $F = m \cdot a = m \cdot \frac{d^2 s}{dt^2}$ (Force = Mass × Acceleration)

Hooke's Law of Spring: $f(x) = k \cdot x$. k is the spring constant.

- Work Done by constant Force.

e.g. 1 How much work is done in lifting a 20-lb weight 6ft off the ground?

$\boxed{W = F \cdot d} = 20 \cdot 6 = 120$ ft-lb.

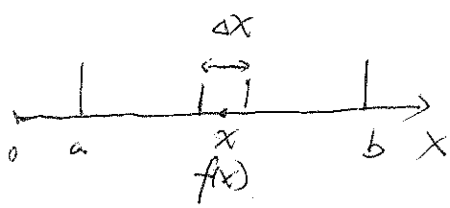


Remark: unit of work is ft-lb.

lb in this problem is unit for force (not for mass).

If one use Newton for force and meter for distance, the unit of work will be J (joule).

- The force is a function of position x, $f(x)$.



The total work equals the sum of these small amount of work over the path.

$W \approx \sum f(x) \cdot \Delta x$.

$\boxed{\text{Work} = \int_a^b f(x) \cdot dx.}$ work done by moving a particle from a to b with force $f(x)$

e.g.2 (S16. mid1). A variable force of $x^2 - 2x$ pounds moves an object along a straight line when it is x feet from the origin. Calculate the work W done in moving the object from $x=2$ to $x=3$ feet.

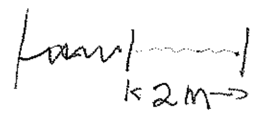
sln: $W = \int_2^3 x^2 - 2x \, dx = \frac{1}{3}x^3 - x^2 \Big|_2^3 = \frac{1}{3} \cdot 3^3 - 3^2 - (\frac{1}{3} \cdot 2^3 - 2^2) = \boxed{\frac{4}{3} \text{ ft-lbs}}$

- One way to get a variable force $f(x)$ is through Hooke's Law.

e.g. 3 (f14, mid1). To hold a spring stretched 2m beyond its natural length requires a force of 12 Newtons. Compute the work needed to stretch the spring from 2m beyond its natural length to 3m beyond its natural length. (in joules).

s/n: Step 1: find spring constant k via given data.

$f(x) = k \cdot x$ (Hooke's law).



plug in $x = 2$, $f = 12$. i.e.

$f(2) = k \cdot 2 = 12 \Rightarrow k = 6$. Therefore, $f(x) = 6 \cdot x$.

Step 2: compute work via formula ②.

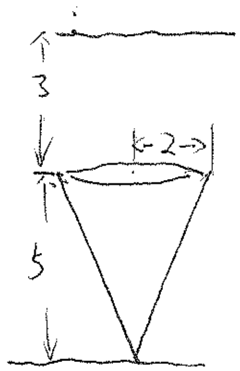
$$W = \int_2^3 f(x)\,dx = \int_2^3 6x\,dx = 6 \cdot \frac{1}{2} x^2 \Big|_2^3 = 3x^2 \Big|_2^3$$

$$= 3 \cdot 3^2 - 3 \cdot 2^2 = \boxed{15 \text{ J}} \text{ (joules)}$$

☆☆ In the following examples, the force is no longer given explicitly (as a function of x)

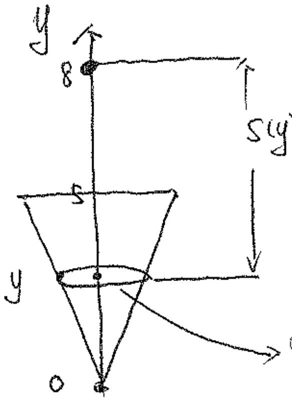
- Work against gravity: $\boxed{\text{Water-Pumping}}$, Cable-lifting.

☆☆ e.g. 4. (s15) A tank is in the shape of a downward-pointing cone (inverted circular cone) which has height 5 feet and radius 2 feet. The tank is full of oil weighing 7 lb/ft³. Find the work it would take to pump the oil from the tank to an outlet 3 feet above the top of the tank.



Idea: Imagine the tank full of oil is a "solid". Cut it into horizontal slice. The total work needed will be the SUM of the work lifting each slice, which eventually leads to the formula.

$$\boxed{W = \int_a^b \sigma \cdot s(y) \cdot A(y)\,dy}$$



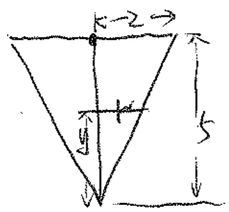
$s(y)$: distance between 'slice' at y and the aim height.

σ: ~~Water~~ Density (water, oil, etc)

$A(y)$: The area of the 'slice' at y. (Area of the cross-section)

sln: $\sigma = 7$ (lb/ft^3), $s(y) = 8 - y$.

(The rest work is to find $A(y)$ from similar triangles)

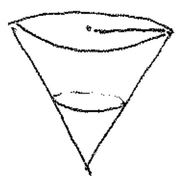


$$\frac{r}{y} = \frac{2}{5} \Rightarrow r = \frac{2}{5}y \Rightarrow A(y) = \pi \cdot r^2 = \pi \cdot \left(\frac{2}{5}y\right)^2$$

According to the formula,
$$W = \int_0^5 7 \cdot (8-y) \cdot \pi \cdot \left(\frac{2}{5}y\right)^2 \cdot dy$$
$$= \int_0^5 7 \cdot (8-y) \cdot \pi \cdot \frac{4}{25} \cdot y^2 \cdot dy$$
$$= \frac{28\pi}{25} \int_0^5 (8-y) \cdot y^2\, dy$$
$$= \frac{28}{25}\pi \cdot \int_0^5 8y^2 - y^3\, dy = \frac{28\pi}{25}\left[\frac{8}{3}y^3 - \frac{1}{4}y^4\right]\Big|_0^5$$
$$= \boxed{\frac{28\pi}{25} \cdot \left[\frac{8}{3} \cdot 5^3 - \frac{1}{4} \cdot 5^4\right]} \text{ (ft·lb)}$$

Remark: The most common types of tanks:

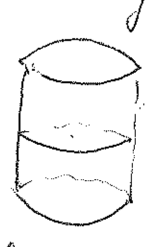
- Inverted circular cone



$A(y) = \pi \cdot r^2$

r depends on y, and can be found via similar triangles.

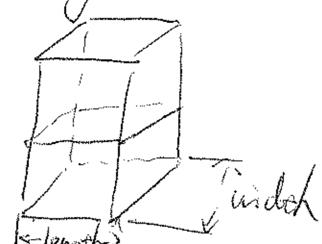
- Vertical Cylinder



$A(y) = \pi \cdot r^2$

r is a constant. (independent of y)

- Rectangular cuboid

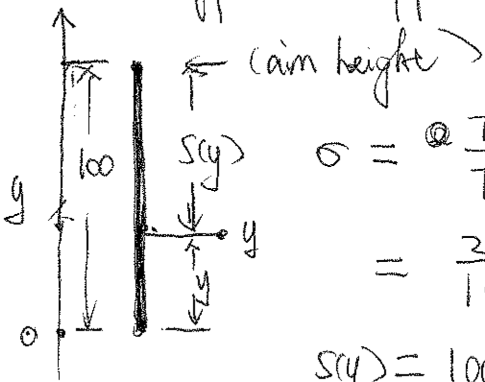


$A(y)$ = length × width (independent of y).

☆ eg.5 (Cable-lifting ⇒ eg.4 in textbook, Page 370)

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building.

soln: (A different approach compared with the solution in the textbook).



$$\sigma = \frac{\text{Total weight}}{\text{Total length}} \quad \text{(density per length)}.$$

$$= \frac{200}{100} = 2 \text{ lb/ft}.$$

$S(y) = 100 - y$. ft

$A(y) = 1$. (☆ unit area since the density is given per length

$$W = \int_0^{100} 2\cdot(100-y)\cdot 1\cdot dy = \int_0^{100} 200 - 2y \cdot dy = 200\cdot y - y^2 \Big|_0^{100}$$

$$= 200\times 100 - 100^2$$

$$= 10{,}000 \text{ ft-lb}.$$

Question: Prove that $\int_0^{100} 2(100-y)\,dy = \int_0^{100} 2y\,dy$ and compare the answer with the expression in the text book.

Hint: U-Sub : $u = 100 - y$.

Answer: set $u = 100 - y$. Then $du = -dy$, $dy = -du$.

$$\int_0^{100} 2(100-y)\,dy = \int_{100}^{0} 2\cdot u\cdot(-du) = -\int_{100}^{0} 2u\,du = \int_0^{100} 2u\cdot du = \int_0^{100} 2y\,dy$$

(Here we use the property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$)